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Some spaces of Summability for Triple Sequences of Fuzzy Numbers

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ABSTRACT: In this study, I extend classical concepts to triple sequences of fuzzy numbers, represented by $X = X_{\mathcal{K},\sigma,\aleph}$ in order to give a fresh framework for understanding summability spaces. Inspired by the growing interest in multidimensional sequence spaces and the use of fuzzy analysis in uncertain settings, I provide a generalized triple difference operator $D_{w}^{\rho}(x)$ which is the basis for constructing a number of novel forms of summability and convergence. Specifically, I study statistically convergent sequences, lacunary statistically convergent sequences, $D_{w}^{\rho}(x)$ -lacunary strongly summable sequences, and $D_{w}^{\rho}(x)$-Cesàro strongly summable sequences in the triple fuzzy scenario. I build a number of inclusion relations between the related function spaces, which are constructed and designated appropriately. Additionally, I demonstrate that some of these spaces are complete metric spaces, expanding previous findings in the theory of fuzzy number sequences to a more flexible and broad framework. The findings of this work not only generalize existing summability techniques but also lay the groundwork for future research in multidimensional sequence analysis and fuzzy approximation.

KEYWORDS: Triple sequences, fuzzy numbers, generalized difference operator $D_{w}^{\rho}(x)$lacunary summability, Cesàro summability, statistical convergence, completeness.

INTRODUCTION

Functional analysis, approximation theory, and decision science are just a few of the disciplines that have been greatly impacted by Zadeh's [1] theory of fuzzy sets. Matloka [4] and Nanda [5] expanded classical sequence analysis utilizing fuzzy numbers by studying the metric completeness, convergence, and boundedness of fuzzy-valued sequences in one dimension.

These concepts were expanded upon in later studies, such as those by Kizmaz [3], Et and Çolak [2], Tripathy [8,9], and others, by adding summability techniques and difference operators. Nevertheless, these investigations were limited to fuzzy number sequences that were single-indexed.

By presenting and analyzing triple-indexed sequences of fuzzy numbers, or sequences of the form $X = X_{\mathcal{K},\sigma,\aleph}$ I hope to generalize these traditional ideas in this work. When dealing with three-dimensional data, this structure naturally emerges in a variety of mathematical and practical applications.

To formalize summability over such structures, I define a generalized triple difference operator $D_{w}^{\rho}(x)$, involving:

- a weight sequence $w = (w_{\mathcal{K},m,n})$,
- a control parameter $\rho \in R$,

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• and a difference order $x \in \mathcal{N}$.

I define strong lacunary summability, Cesàro summability, and statistical convergence for fuzzy triple sequences using this operator. After that, I build the corresponding summability spaces, examine their connections, and demonstrate a number of structural and metric features, including completeness. In addition to generalizing well-known findings, this work presents a versatile framework for additional research in fuzzy sequence analysis across multiple domains.

2. Preliminaries and Definitions

Let $\mathbb{D}$ denote the set of all closed and bounded intervals on R.

For $\mathcal{A} = [\mathcal{A}_1, \mathcal{A}_2]$, $\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2]$ in $\mathbb{D}$, we define:

• Partial order: $\mathcal{A} \leq \mathcal{B}$ if $\mathcal{A}_1 \leq \mathcal{B}_1$ and $\mathcal{A}_2 \leq \mathcal{B}_2$

• Metric: $h(\mathcal{A}, \mathcal{B}) = \max\{|\mathcal{A}_1 - \mathcal{B}_1|, |\mathcal{A}_2 - \mathcal{B}_2|\}$.

Then $(\mathbb{D}, h)$ forms a complete metric space**.**

A fuzzy number is a fuzzy subset of $\Re$ which is normal, convex, upper semi-continuous, and has compact support. The set of all such fuzzy numbers is denoted by $L(R)$ For $\mathfrak{M} \in L(R)$, the α-cut is:

$$\mathfrak{M}_\alpha = \{ t \in R : \mu_X(t) \geq \alpha \}, \qquad \alpha \in [0,1].$$

Define a metric on $L(\Re)$ as:

$$d(\mathfrak{M}, \mathfrak{N}) = \sup_{0 \leq \alpha \leq 1} h(\mathfrak{M}_\alpha, \mathfrak{N}_\alpha)$$

which makes $(L(R), d)$ a complete metric space.

2.1: Definition (Triple Fuzzy Sequence)

A mapping $X = (X_{\kappa,\sigma,\aleph}): \mathbb{N}^3 \to L(R)$, where each element $X_{\kappa,\sigma,\aleph}$ is a fuzzy number, is a triple sequence of fuzzy numbers. $\mathbb{f}_\mathcal{T}$ represents the set of all such sequences.

2.2: Definition (Convergence)

A sequence $X = (X_{\kappa,\sigma,\aleph}) \in \mathbb{f}_\mathcal{T}$ is said to converge $t_0 X_0 \in L(R)$, if:

$$d(X_{\kappa,\sigma,\aleph}, X_0) = 0, \qquad lim\, \kappa, \sigma, \aleph \to \infty .$$

2.3: Definition 2.3 (Cauchy)

$X = (X_{\kappa,\sigma,\aleph})$ is called Cauchy if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\kappa, \sigma, \aleph, \acute{\kappa}, \acute{\sigma}, \acute{\aleph} > N$, we have: $d(X_{\kappa,\sigma,\aleph}, X_{\acute{\kappa},\acute{\sigma},\acute{\aleph}}) < \epsilon$.

2.4: Definition (Boundedness)

The sequence $X = (X_{\kappa,\sigma,\aleph}) \in \mathbb{f}_\mathcal{T}$ is said to be bounded if there exists $\mho \in L(R)$, such that: $d(X_{\kappa,\sigma,\aleph}, 0) \leq d(\mho, 0)$ for all $\kappa, \sigma, \aleph \in \mathbb{N}$

2.5: Definition (Triple Difference Operator)

Let $\mathcal{w} = (\mathcal{w}_{\kappa,\sigma,\aleph})$ be a non-zero weight sequence, $\rho \in R$, and $x \in \mathbb{N}$. The generalized triple difference operator $D^\rho_\mathcal{w}(x)$ applied to $X = (X_{\kappa,\sigma,\aleph})$ is defined as:

$$D^\rho_\mathcal{w}(x) X_{\kappa,\sigma,\aleph} = \sum_{i=0}^{x}\sum_{j=0}^{x}\sum_{l=0}^{x}(-1)^{i+j+l}\binom{x}{i}\binom{x}{j}\binom{x}{l}\mathcal{w}^\rho_{\kappa-i,\sigma-j,\aleph-l} X_{\kappa,\sigma,\aleph}$$

with the assumption that $X_{a,b,c} = \bar{0}$ and $\mathcal{w}_{a,b,c} = 0$ whenever any index is non-positive.

Remark2.1:

The above-defined operator $D^\rho_\mathcal{w}(x)$ is a natural extension of the conventional one-dimensional difference operator Δv^{rs}, which was first described in [6] and extensively examined in [16,17]. The suggested operator, which has a multi-dimensional structure and weighted control, is designed to act on triple-indexed sequences of fuzzy numbers, whereas the traditional version works on single-indexed sequences. This generalization lays the groundwork for the triple sequence spaces examined in this research and permits a richer class of summability behaviors in multi-indexed fuzzy environments.

3. Main Concepts and New Spaces

Throughout this section, Let $\theta = (q_p)$ be a lacunary sequence of positive integers such that $q_0 = 0$, $q_p < q_{p+1}$, and define the intervals $I_p = (q_{p-1}, q_p]$. We denote $t_p = q_p - q_{p-1}$ and assume $\lim_{p\to\infty} t_p = \infty$.

3.1: (Triple Lacunary Strong Summability)

A triple fuzzy sequence $X = (X_{\kappa,\sigma,\aleph}) \in \mathbb{f}_\mathcal{T}$ is said to be lacunary strongly summable to a fuzzy number $x_0 \in L(R)$, with respect to the operator $D^\rho_\mathcal{w}(x)$ if there exists a fuzzy number $x_0 \in L(R)$, such that:

$$\lim_{P\to\infty} \frac{1}{t_p^3} \sum_{(\kappa,\sigma,\aleph)\in I_p^3}^{N} d(D^\rho_\mathcal{w}(x)\, X_{\kappa,\sigma,\aleph}, X_0) = 0$$

The set of all such sequences is denoted by: $\mathbb{f}_\mathcal{T}[\mathcal{w}, D^\rho_\mathcal{w}(x)]^L_\theta$

3.2: Definition (Triple Cesàro Strong Summability)

If there exists $x_0 \in L(R)$, then a sequence $X = (X_{\kappa,\sigma,\aleph}) \in \mathbb{f}_\mathcal{T}$ is considered Cesàro highly summable with respect to $D^\rho_\mathcal{w}(x)$.

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A sequence $X=(X_{\varkappa,\sigma,\aleph}) \in \mathbb{f}_{\mathcal{T}}$ is said to be Cesàro strongly summable with respect to $D_w^{\rho}(x)$ if there exists $x_0 \in L(R)$, such that:

$$\lim_{N\to\infty}\frac{1}{N^3}\sum_{\varkappa,\sigma,\aleph=1}^{N} d\left(D_w^{\rho}(x)X_{\varkappa,\sigma,\aleph},X_0\right)=0.$$

We denote this space by: $\mathbb{f}_{\mathcal{T}}\left[w,D_w^{\rho}(x)\right]_{\rho}^{L}$.

3.3: Definition (Triple Statistical Convergence)

A sequence $X=(X_{\varkappa,\sigma,\aleph}) \in \mathbb{f}_{\mathcal{T}}$ is said to be Statistically Convergence to $x_0 \in L(R)$, with respect to $D_w^{\rho}(x)$ if for every $\epsilon>0$, we have:

$$\lim_{N\to\infty}\frac{1}{N^3}n\left\{(\varkappa,\sigma,\aleph)\le N: d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},X_0\right)\ge\epsilon\right\}=0.$$

where n denotes the cardinality of the set. We denote this space by: $\mathbb{f}_{\mathcal{T}}\left[w,D_w^{\rho}(x)\right]_{S}^{L}$.

3.4: Definition (Triple Lacunary Statistical Convergence)

A sequence $X=(X_{\varkappa,\sigma,\aleph}) \in \mathbb{f}_{\mathcal{T}}$ is said to be Statistically Convergence to $x_0 \in L(R)$, if for every $\epsilon>0$:

$$\lim_{p\to\infty}\frac{1}{t_p^3}n\left\{(\varkappa,\sigma,\aleph)\in I_p^3: d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},X_0\right)\ge\epsilon\right\}=0.$$

We denote this space by: $\mathbb{f}_{\mathcal{T}}\left[w,D_w^{\rho}(x)\right]_{S_\theta}^{L}$.

4.Theorems and Structural Results

Theorem4.1:

Let $\theta=(q_p)$ be lacunary Sequence and if $X=(X_{\varkappa,\sigma,\aleph}) \in \mathbb{f}_{\mathcal{T}}\left[w,D_w^{\rho}(x)\right]_{\theta}^{L}$

Then $X \in \mathbb{f}_{\mathcal{T}}\left[w,D_w^{\rho}(x)\right]_{S_\theta}^{L}$.

Proof:

The number of sites where the distance exceeds any ε > 0 must decrease to zero proportionately since the average of distances tends to zero for each lacunary block I_p^3. In fact, for any fixed ε > 0:

$$\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},X_0\right) \ge \frac{1}{t_p^3}\sum_{\substack{(\varkappa,\sigma,\aleph)\in I_p^3\\ d\ge\epsilon}} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},X_0\right)$$

$$\ge \epsilon\frac{1}{t_p^3}n\left\{(\varkappa,\sigma,\aleph)\in I_p^3: d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},X_0\right)\ge\epsilon\right\}.$$

Considering the limit as $p\to\infty$, the lacunary statistical percentage must likewise approach 0 since the left-hand side approaches 0 by hypothesis.

Theorem4.2:

If $X \in \mathbb{f}_{\mathcal{T}}$ is bounded with respect to $D_w^{\rho}(x)$,and $X \in \mathbb{f}_{\mathcal{T}}\left[w,D_w^{\rho}(x)\right]_{S}^{L}$, then $X \in \mathbb{f}_{\mathcal{T}}\left[w,D_w^{\rho}(x)\right]_{\rho}^{L}$.

Proof:

Since $X \in D_{w\rho}^{(x)}s$, there exists a fuzzy number $x_0 \in L(R)$such that for every $\epsilon>0$, we have:

$$\lim_{N\to\infty}\frac{1}{N^3}\left|(\varkappa,\sigma,\aleph)\le N: d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)\ge\epsilon\right| = 0$$

Assume that the sequence $D_w^{\rho}(x)X_{\varkappa,\sigma,\aleph}$ is bounded. Then there exists a constant M > 0 such that for all $\varkappa,\sigma,\aleph\in N$, we have:

$$d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)\le M$$

To analyze the full Cesàro average, we consider:

$$\frac{1}{N^3}\sum_{\varkappa=1}^{N}\sum_{\sigma=1}^{N}\sum_{\aleph=1}^{N} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)$$

Define the set:

$$A_N^{\varepsilon}\left\{(\varkappa,\sigma,\aleph)\in[1,N]^3N: d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)\ge\epsilon\right\}$$

Then we split the total sum into two parts:

$$\frac{1}{N^3}\sum_{(\varkappa,\sigma,\aleph)\in A_N^{\varepsilon}}^{N} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right) + \sum_{(\varkappa,\sigma,\aleph)\notin A_N^{\varepsilon}}^{N} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)$$

Since $d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)\le M$, we have:

$$\frac{1}{N^3}\sum_{A_N^{\varepsilon}}^{N} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right).M,$$

But by statistical convergence, $\frac{|A_N^{\varepsilon}|}{N^3}\to 0$.

For the second part, every term is less than ε, so:

$$\frac{1}{N^3}\sum_{A_N^{\varepsilon}}^{N} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)\le\varepsilon,$$

Hence:

$$\frac{1}{N^3}\sum_{\varkappa=1}^{N}\sum_{\sigma=1}^{N}\sum_{\aleph=1}^{N} d\left(D_w^{\rho}(x)\,X_{\varkappa,\sigma,\aleph},x_0\right)\le M.\frac{|A_N^{\varepsilon}|}{N^3}+\varepsilon,$$

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Taking the limit as $N\left(\frac{1}{N^3}\sum_{\varkappa=1}^{N}\sum_{\sigma=1}^{N}\sum_{\aleph=1}^{N} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, \mathrm{x}_0\right)=0\right)$,, and using $\varepsilon > 0$ arbitrary, we conclude:

$$\lim_{N\to\infty}\left(\frac{1}{N^3}\sum_{\varkappa=1}^{N}\sum_{\sigma=1}^{N}\sum_{\aleph=1}^{N} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, \mathrm{x}_0\right)=0\right),$$

Thus,

$X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\rho}^{L}$.

Theorem 4.3:
Let $\theta = q_p$ be a lacunary sequence. If a triple fuzzy sequence $X = \left(X_{\varkappa,\sigma,\aleph}\right) \in \mathbb{f}_{\mathcal{T}}$ is bounded with respect to the operator $D_w^{\rho}(x)$ and is lacunary statistically convergent to X_0(i.e., $X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{S_\theta}^{L}$), then it is lacunary strongly summable to X_0, meaning $X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L}$.

Proof:
Let $X_0 \in \mathrm{L}(R)$ be the lacunary statistical limit of X, so for all $\epsilon > 0$, we have:

$$\lim_{p\to\infty}\frac{1}{t_p^3}\left|\left\{(\varkappa,\sigma,\aleph)\in I_p^3 : d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right)\geq\epsilon\right\}\right|$$

Now, assume that there exists M > 0 such that:

$$d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right) \leq \mathrm{M} \text{ for all } \varkappa,\sigma,\aleph.$$

Define the set

$$A_p^{\epsilon} = \left\{(\varkappa,\sigma,\aleph)\in I_p^3 : d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right)\geq\epsilon\right\}.$$

Then we have:

$$\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right)\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right),$$

We estimate each term:

$$\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right) \leq \frac{\left|A_p^{\epsilon}\right|}{t_p^3}.\mathrm{M},$$

$$\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3 / A_p^{\epsilon}} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right) \leq \epsilon,$$

Hence:

$$\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right) \leq \frac{\left|A_p^{\epsilon}\right|}{t_p^3}.\mathrm{M} + \epsilon,$$

Taking the limit as $p \to \infty$, we obtain:

$$\lim_{p\to\infty} sup\,\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right) \leq \epsilon$$

Since $\epsilon > 0$ is arbitrary, it follows that:

$$\lim_{p\to\infty} -\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right) = 0$$

Thus $X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L}$.

Theorem4.4: (Equivalence)
Let $X \in \mathbb{f}_{\mathcal{T}}$ is $D_w^{\rho}(x)$- bounded. Then: $X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{S_\theta}^{L} \Leftrightarrow X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L}$.

Proof:
Follows directly by combining Theorems **4.1** and **4.3**.
Let $\theta = \left(q_p\right)$ be lacunary Sequence and let $X = \left(X_{\varkappa,\sigma,\aleph}\right) \in \mathbb{f}_{\mathcal{T}}$ be such that:

$$\lim_{p\to\infty}\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3}^{N} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, X_0\right) = 0,$$

Then X is lacunary Statistically Convergence to X_0.

Theorem4.5:
Let $\theta = \left(q_p\right)$ be lacunary Sequence and d, and define the metric
$G: \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L} \times \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L} \to R$ by:

$$G(X,Y) = \sup_{p\epsilon N}\left(\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^{\rho}(x)\, X_{\varkappa,\sigma,\aleph}, D_w^{\rho}(x)\, Y_{\varkappa,\sigma,\aleph,}\right)\right)$$

Then $\left(\mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L}, G\right)$ is a complete metric space.

Proof:
Let $\left\{x^i\right\}$ $i \in N$ be a Cauchy sequence in $\mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L}$. Then for every $\varepsilon > 0$, there exists $n_0 \in N$ such that for all i, j $\geq n_0$:

$$\left(\mathrm{X}^i, \mathrm{Y}^j\right) < \varepsilon.$$

this implies that for all p $\in N$:

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$$\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, D_w^\rho(x)\,Y_{\varkappa,\sigma,\aleph}\right) < \varepsilon.$$

Thus, for each fixed triple $(\varkappa,\sigma,\aleph)\in N^3$, the sequence: $D_w^\rho(x)X_{\varkappa,\sigma,\aleph}, n^i)i$
is a Cauchy sequence in the complete metric space L(ℜ), hence convergent. Define:

$$Y_{\varkappa,\sigma,\aleph} = \lim_{i\to\infty} D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}\ .$$

We now define $X=(X_{\varkappa,\sigma,\aleph})$such that $D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph} = Y_{\varkappa,\sigma,\aleph}$. It remains to show that $X\in \mathbb{f}_{\mathcal{T}}\left[w, D_w^\rho(x)\right]_\theta^L$. and that $X^i\to X$under the metric G.

Using the triangle inequality and passing to the limit, we get:

$$G(X^i,X) = \sup_p\left(\frac{1}{t_p^3}\sum_{(\varkappa,\sigma,\aleph)\in I_p^3} d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, Y_{\varkappa,\sigma,\aleph,}\right)\right) \to 0.$$

Hence, $X\in \mathbb{f}_{\mathcal{T}}\left[w, D_w^\rho(x)\right]_\theta^L$, and the space is complete.

Special Case

Assume we have a constant weight sequence $w_{\mathcal{K},m,n}=1$ and $X=1$ The operator reduces to a first-order backward difference in three directions under these circumstances:

$$D^1X_{\varkappa,\sigma,\aleph} = X_{\varkappa,\sigma,\aleph} - X_{\varkappa-1,\sigma,\aleph} - X_{\varkappa,\sigma-1,\aleph} - X_{\varkappa,\sigma,\aleph-1} + X_{\varkappa-1,\sigma-1,\aleph-1}.$$

This shows that local stability of the sequence over adjacent indices in the 3D lattice correlates to convergence in $\mathbb{f}^1L_\theta$. Classical one-dimensional difference spaces are then directly generalized by the completeness argument.

Theorem 4.6:

Let $\theta=(q_p)$ be a lacunary sequence satisfying: $\lim_{p\to\infty} inf\,\frac{q_p}{q_{p-1}} > 1$ If a triple fuzzy sequence $X=(X_{\varkappa,\sigma,\aleph})\in \mathbb{f}_{\mathcal{T}}$ is statistically convergent to X_0 with respect to the operator $D_w^\rho(x)$ (i.e., $X\in \mathbb{f}_{\mathcal{T}}\left[w, D_w^\rho(x)\right]_s^L$), then it is also lacunary statistically convergent to X_0, meaning $X\in \mathbb{f}_{\mathcal{T}}\left[w, D_w^\rho(x)\right]_{s_\theta}^L$

Proof:

Let $X\in \mathbb{f}_{\mathcal{T}}\left[w, D_w^\rho(x)\right]_s^L$. By the given condition $\lim_{p\to\infty} inf\,\frac{q_p}{q_{p-1}} > 1$, there exists a $\delta>0$, such that for sufficiently large p we have: $\frac{q_p}{q_{p-1}} > 1+\delta$

This inequality can be rewritten in terms of the block length $t_p = q_p - q_{p-1}$ as:

$$\frac{q_p}{t_p} \le \frac{1+\delta}{\delta}$$

The relationship between the index set of the lacunary block $I_p^3 = I_p\times I_p\times I_p$and the entire three-dimensional grid $\{1,2,\dots,q_p\}^3$is now examined for any ϵ>0. The number of elements in the block that violate the convergence threshold is bounded above by the total number of such elements up to q_p:

$$\left|\{(\varkappa,\sigma,\aleph)\le I_p^3 : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\}\right| \le \left|\{(\varkappa,\sigma,\aleph)\le q_p : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\}\right|,$$

Dividing both sides by t_p^3, we obtain:

$$\frac{1}{t_p^3}\left|\{(\varkappa,\sigma,\aleph)\in I_p^3 : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\}\right| \le \frac{1}{t_p^3}\left|\{(\varkappa,\sigma,\aleph) \le q_p : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\}\right|.$$

$$\le \left(\frac{q_p}{t_p}\right)^3 \frac{1}{q_p^3}\left\{\left|(\varkappa,\sigma,\aleph)\le q_p : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\right|\right\}$$

$$\le \left(\frac{1+\delta}{\delta}\right)^3 \frac{1}{q_p^3}\left\{\left|(\varkappa,\sigma,\aleph)\le q_p : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\right|\right\}$$

The expression $\left(\frac{1+\delta}{\delta}\right)^3$remains a limited constant when the limit is taken as $p\to\infty$ on both sides. However, because X is statistically convergent to X_0 globally (i.e., the natural density approaches to zero as $q_p\to\infty$), the factor:

$$\lim_{p\to\infty}\frac{1}{q_p^3}\left\{\left|(\varkappa,\sigma,\aleph)\le q_p : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\right|\right\} = 0$$

Therefore, the entire right-hand side product approaches zero:

$$\lim_{p\to\infty}\frac{1}{t_p^3}\left\{\left|(\varkappa,\sigma,\aleph)\in I_p^3 : d\left(D_w^\rho(x)\,X_{\varkappa,\sigma,\aleph}, X_0\right)\ge\epsilon\right|\right\} = 0$$

This proves that $X\in \mathbb{f}_{\mathcal{T}}\left[w, D_w^\rho(x)\right]_{s_\theta}^L$.

Special Case (Sharpness of the Condition)

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Given $q_p = 2p$, so $t_p = q_p - q_{p-1} = 2$. The necessary condition is thus broken: $\frac{t_p}{p} = \frac{2}{p} \to 0$,

A sequence that is statistically convergent but not lacunary statistically convergent can be created in such a situation. Therefore, the theorem cannot hold unless $\lim inf \frac{t_p}{p} > 0$.

Theorem 4.7:

If the lacunary sequence increases too slowly, statistical convergence does not necessarily imply lacunary statistical convergence. Consider a lacunary sequence $\theta=(q_p)$that satisfies

$$\lim_{p\to\infty} inf \frac{q_p}{q_{p-1}} = 1.$$

Then there exists a triple fuzzy sequence $X = (X_{\varkappa,\sigma,\aleph}) \in \mathbb{f}_{\mathcal{T}}$ such that:

$X \in \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{S}$ but $X \notin \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{S_\theta}$

Proof:

Let $A \in \mathrm{L}(R)$ be a fixed non-zero fuzzy number whose distance from the fuzzy zero is strictly positive, meaning: $d(A, \bar{0}) > 0$,

We define our sequence X through its generalized triple difference operator $D^{\rho}_{w}(x)\, X_{\varkappa,\sigma,\aleph}$, as follows:

$D^{\rho}_{w}(x)\, X_{\varkappa,\sigma,\aleph}= A$,if $\varkappa = \sigma, \aleph = q_p$ for all $p \in N$

$D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph} = \bar{0}$, otherwise

First, we confirm that this sequence has global statistical convergence to $\bar{0}$. The number of non-zero terms in a three-dimensional cube of size N^3 is at most equal to the number of lacunary indices q_p that are less than or equal to N for any given threshold $\epsilon > 0$ set such that $d(A, \bar{0}) > \epsilon$.

Thus, the cardinality of this exceptional index set is bounded above by N:

$$\left|\{(\varkappa, \sigma, \aleph) \le N : d\left(D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph}, \bar{0}\right) \ge \epsilon\}\right| \le N$$

Now, we calculate the natural density of this exceptional set by dividing by N^3 and taking the limit as $N \to \infty$:

$$\lim_{N\to\infty} \frac{1}{N^3} \left|\{(\varkappa, \sigma, \aleph) \le N : d\left(D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph}, \bar{0}\right) \ge \epsilon\}\right| \le \lim_{N\to\infty} \frac{N}{N^3} \le \lim_{N\to\infty} \frac{1}{N^2} = 0\,.$$

Since the limit of the density is identically zero, the sequence X is globally statistically convergent to $\bar{0}$. Therefore: $X \in \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{S}$ Next, we evaluate the lacunary statistical convergence within the three-dimensional block intervals I^3_p. By our definition of the sequence, for each block index p, the term at the upper boundary where $\varkappa = \sigma, \aleph = q_p$is exactly equal to A.

Therefore, within any lacunary block I^3_p, there is always at least one element where the distance condition is violated:

$$\left|\{(\varkappa, \sigma, \aleph) \in I^3_p : d\left(D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph}, \bar{0}\right) \ge \epsilon\}\right| \ge 1$$

Under the hypothesis that $\lim_{p\to\infty} inf \frac{q_p}{q_{p-1}} = 1$, the length of the blocks $t_p = q_p - q_{p-1}$ grows too slowly relative to q_p, which implies that the fraction $\frac{1}{t_p^3}$does not approach zero as $p \to \infty$.

Consequently, the block density fails to vanish:

$$\lim_{p\to\infty} \frac{1}{t_p^3} \left|\{(\varkappa, \sigma, \aleph) \in I^3_p : d\left(D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph}, \bar{0}\right) \ge \epsilon\}\right| \nrightarrow 0.$$

This directly implies that the sequence is not lacunary statistically convergent to $\bar{0}$, meaning: $X \notin \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{S_\theta}$.

Theorem 4.8:

Let $X = (X_{\varkappa,\sigma,\aleph}) \in \mathbb{f}_{\mathcal{T}}$, if there exists a fuzzy number $X_0 \in \mathrm{L}(\Re)$ and a positive integer $N_0 \in \mathbb{N}$ such that:

$$D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph} = X_0 \quad \text{for all } \varkappa, \sigma, \aleph > N_0.$$

Then:

$$X \in \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{\theta} \cap \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{\sigma} \cap \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{S} \cap \mathbb{f}_{\mathcal{T}}[w, D^{\rho}_{w}(x)]^{L}_{S_\theta}.$$

Proof:

By hypothesis, the difference operator $D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph}$ becomes identically equal to X_0 for all indices strictly greater than N_0,Consequently, the distance metric satisfies:

$$d\left(D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph}, X_0\right) = d(X_0, X_0) = 0, \quad \text{for all } \varkappa, \sigma, \aleph > N_0$$

This implies that the number of terms for which $d\left(D^{\rho}_{w}(x) X_{\varkappa,\sigma,\aleph}, X_0\right) = 0$, is strictly finite and bounded by the dimension N_0^3.

Any finite number of non-zero terms contributes zero to the final limits because all four convergence and summability definitions (Strong Lacunary, Strong Cesàro, Statistical, and Lacunary Statistical) evaluate the asymptotic limits or natural densities as the grid size approaches infinity (∞). In particular:

1. For Cesàro and Lacunary Strong Summability: The finite sum of distances divided by by N^3 or t_p^3 vanishes to 0 as $N \to \infty$ or as $p \to \infty$.

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2. For Statistical and Lacunary Statistical Convergence: The cardinality of the exceptional index set containing non-zero distances is finite, meaning its natural density is identically 0 in the limit. Therefore, all four limits converge to zero simultaneously, which structurally forces X to belong to each of the four spaces.

Therefore, all four limits converge to zero simultaneously, which structurally forces X to belong to each of the four spaces.

Theorem 4.9:

Let θ=$q_p$be a lacunary Sequence If a triple fuzzy sequence satisfies $\limsup_{p\to\infty} \frac{q_p}{q_{p-1}} < \infty$,Then:

$$\mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\sigma}^{L} \subset \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L}$$

Proof:

Let θ=(q_p)be a lacunary Sequence be a lacunary sequence, and let $I_p = (q_{p-1}, q_p], t_p = q_p - q_{p-1}$ We aim to show that if $X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\sigma}^{L}$ Then:

$$\lim_{p\to\infty} \frac{1}{t_p^3} \sum_{(\varkappa,\sigma,\aleph)\in I_p^3}^{N} d\left(D_w^{\rho}(x) X_{\varkappa,\sigma,\aleph}, X_0\right) = 0$$

Since the three-dimensional lacunary block interval $I_p^3$is strictly contained within the full grid $\{1,2,\dots,q_p\}^3$, the summation over I_p^3 is bounded above by the total sum up to q_p:

$$\frac{1}{t_p^3} \sum_{(\varkappa,\sigma,\aleph)\in I_p^3}^{N} d\left(D_w^{\rho}(x) X_{\varkappa,\sigma,\aleph}, X_0\right) \leq \frac{1}{t_p^3} \sum_{\varkappa=1}^{q_p} \sum_{\sigma=1}^{q_p} \sum_{\aleph=1}^{q_p} d\left(D_w^{\rho}(x) X_{\varkappa,\sigma,\aleph}, X_0\right)$$

By multiplying and dividing the right-hand side by q_p^3, we can restructure the inequality as follows:

$$\leq \left(\frac{q_p}{t_p}\right)^3 \frac{1}{q_p^3} \sum_{\varkappa=1}^{q_p} \sum_{\sigma=1}^{q_p} \sum_{\aleph=1}^{q_p} d\left(D_w^{\rho}(x) X_{\varkappa,\sigma,\aleph}, X_0\right)$$

Under the given condition $\limsup_{p\to\infty} \frac{q_p}{q_{p-1}} < \infty$, the term $\frac{q_p}{t_p}$ is uniformally bounded above by a positive constant, say $C > 0$. Therefore, its cubic power is also bounded:

$$\left(\frac{q_p}{t_p}\right)^3 \leq C^3.$$

The second factor tends to 0 due to Cesàro convergence toq_p, tends to 0 as $p \to \infty$ since $X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\sigma}^{L}$ by hypothesis:

$$\lim_{p\to\infty} \frac{1}{q_p^3} \sum_{\varkappa=1}^{q_p} \sum_{\sigma=1}^{q_p} \sum_{\aleph=1}^{q_p} d\left(D_w^{\rho}(x) X_{\varkappa,\sigma,\aleph}, X_0\right) = 0$$

Consequently, the entire product on the right-hand side vanishes in the limit:

$$\lim_{p\to\infty} \frac{1}{t_p^3} \sum_{(\varkappa,\sigma,\aleph)\in I_p^3}^{N} d\left(D_w^{\rho}(x) X_{\varkappa,\sigma,\aleph}, X_0\right) = 0$$

Hence, $X \in \mathbb{f}_{\mathcal{T}}\left[w, D_w^{\rho}(x)\right]_{\theta}^{L}$, which completes the proof.

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